On Stability and the Spectrum Determined Growth Condition for Spatially Periodic Systems

Makan Fardad and Bassam Bamieh

***Abstract*— We consider distributed parameter systems where the underlying dynamics are spatially periodic on the real line. We examine the problem of exponential stability, namely whether the semigroup e^{At} decays exponentially in time. It is known that for distributed systems the condition that the spectrum of A belongs to the open left-half plane is, in general, not sufficient for exponential stability. Those systems for which this condition *is* sufficient are said to satisfy the Spectrum Determined Growth Condition (SDGC). In this work we separate A into a spatially invariant operator and a spatially periodic operator. We find conditions for the spatially invariant part to satisfy the SDGC. We then show that the SDGC remains satisfied under the addition of the spatially periodic operator, if this operator is small enough relative to the spatially invariant one. A similar method is used to derive conditions which guarantee that A has left-half plane spectrum, and thus the system is exponentially stable. The results are demonstrated through simple illustrative examples.**

I. Introduction

In all engineering applications, the temporal stability of a system is of central importance. In linear systems theory, assessing exponential stability is of particular interest. For finite-dimensional systems (systems with finite-dimensional state-space), exponential decay of $\|e^{At}\|$ is guaranteed if the spectrum of the A-matrix lies inside the open left-half of the complex plane (open LHP).

The situation is much more complicated in the case of infinite-dimensional systems. For example, it is possible that the spectrum of the A-operator of such a system lies inside the open LHP, and yet $\|e^{At}\|$ actually grows exponentially [1]–[3]. In such cases it is said that the *spectrum-determined growth condition* is not satisfied [3].

Yet there exists quite a wide range of infinite-dimensional systems for which the spectrum-determined growth condition *is* satisfied. These include (but are not limited to) systems for which the A-operator is *sectorial* (also known as an operator which generates a *holomorphic* or *analytic* semigroup) [4]–[6] or is a *Reisz-spectral* operator [7]. In this paper we focus on sectorial operators.

Thus to establish exponential stability of a system, one line of attack would be to show simultaneously that (i) A is sectorial *and* (ii) the spectrum of A lies in the open LHP. But this still does not make the problem trivial. In fact proving that an infinite-dimensional operator is sectorial, and then finding its spectrum, can in general be extremely difficult.

This work is partially supported by AFOSR Grant FA9550-04-1-0207.

M. Fardad and B. Bamieh are with the Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, CA 93105-5070. email: fardad@engineering.ucsb.edu, bamieh@engineering.ucsb.edu.

In this paper we will be dealing with a class of *spatially periodic* systems on the real line. These are systems for which the system-operators A, B, and C are spatially periodic [8] (i.e., they commute only with spatial translations of size equal to some real scalar $\mathrm{X} > 0$ called the period). We consider the A-operator as the sum of a spatially invariant [9] operator A^{o} and a spatially periodic operator $\epsilon\, E$, where ϵ is a complex scalar. Our aim is to find conditions under which this system is exponentially stable.

To show (i) and (ii) we take an indirect route. We first find conditions on the spatially invariant operator A^{o} such that (i) and (ii) are satisfied. We then show that (i) and (ii) will *remain* satisfied if the spatially periodic operator E is "weaker" than A^{o} in a sense that we describe and if ϵ is small enough. The reason for this indirect approach is that (i) and (ii) are much easier to check for a spatially invariant operator than they are for a spatially periodic one. All conditions we derive are in the Fourier domain and can be checked *pointwise* in the spatial-frequency variable $k \in \mathbb{R}$.

Our presentation is organized as follows. We briefly review the frequency representation of spatially periodic operators in Section II. We describe the problem set up in Section III. Section IV deals with general notions of the spectrum and sectorial operators. Section V contains the main contributions of the paper and is divided into two parts; the first part deals with condition (i) described above, and the second part with condition (ii). Conclusions and future directions are given in Section VI. Most proofs and technical material are relegated to the Appendix at the end of the paper.

Notation: We use $k \in \mathbb{R}$ to characterize the spatial-frequency variable, also known as the *wave-number*. $\Sigma(T)$ is the spectrum of the operator T, and $\Sigma_{\mathrm{p}}(T)$ its point spectrum, and $\rho(T)$ its resolvent set. To avoid clutter, we do not index norms on different function/operator spaces. We use $\|\cdot\|$ to denote both function and (induced) operator norms on infinite-dimensional spaces, and the Euclidean norm for finite-dimensional vectors and matrices; the difference will be clear from the context. $\mathbb{C}^-$ denotes the open left-half of the complex plane, and $j := \sqrt{-1}$. $\overline{\mathfrak{S}}$ is the closure of the set $\mathfrak{S} \subset \mathbb{C}$. We may use the same notation for a spatially invariant operator and its Fourier symbol.

II. Preliminaries

In this section we briefly discuss the frequency domain representation of spatially periodic operators. For a detailed account the reader is referred to [8] and [10].

Let $\hat{u}(k)$ and $\hat{y}(k)$ denote the Fourier transforms of two spatial functions $u(x)$ and $y(x)$ respectively. If u and y are

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE **2005**	2. REPORT TYPE	3. DATES COVERED **00-00-2005 to 00-00-2005**

4. TITLE AND SUBTITLE **On Stability abd the Spectrum Determined Growth Condition for Spatially Periodic Systems**	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Department of Electrical and Computer Engineering,University of California,Santa Barbara,CA,93106**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR’S ACRONYM(S) 11. SPONSOR/MONITOR’S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**		**7**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

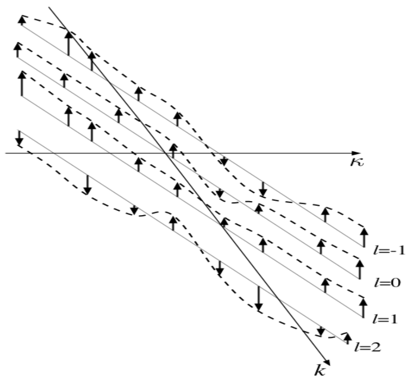

Fig. 1. The frequency kernel representation of a spatially periodic operator G.

related by a linear operator $y = Gu$, we have

$$\begin{aligned} y(x) &= \int_{\mathbb{R}} G(x,\chi)\, u(\chi)\, d\chi \\ &\mathscr{F}_x \updownarrow \\ \hat{y}(k) &= \int_{\mathbb{R}} \hat{G}(k,\kappa)\, \hat{u}(\kappa)\, d\kappa \end{aligned} \tag{1}$$

where G and $\hat{G}$ are *kernel functions* in the spatial and Fourier domain, respectively. It is shown in [8] [10] that the most general spatially periodic operator can be represented in the Fourier domain as an operator with a kernel function of the form

$$\hat{G}(k,\kappa) = \sum_{l\in\mathbb{Z}} \hat{g}_l(k)\; \delta(k-\kappa-\Omega l), \tag{2}$$

where $\hat{g}_l(k)$, for each k, can in general be a matrix. Such a kernel function can be visualized in Figure 1. [8] also describes how (2) can be written as

$$y_\theta = \mathcal{G}_\theta\, u_\theta, \qquad \theta \in [0,\Omega),$$

which for any given θ has a (bi-infinite) matrix representation

$$\begin{bmatrix} \vdots \\ \hat{y}(\theta-\Omega) \\ \hat{y}(\theta) \\ \hat{y}(\theta+\Omega) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & \hat{g}_0(\theta-\Omega) & \hat{g}_{-1}(\theta-\Omega) & \hat{g}_{-2}(\theta-\Omega) & \\ \ddots & \hat{g}_1(\theta) & \hat{g}_0(\theta) & \hat{g}_{-1}(\theta) & \ddots \\ & \hat{g}_2(\theta+\Omega) & \hat{g}_1(\theta+\Omega) & \hat{g}_0(\theta+\Omega) & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}(\theta-\Omega) \\ \hat{u}(\theta) \\ \hat{u}(\theta+\Omega) \\ \vdots \end{bmatrix}. \tag{3}$$

In this setting, $\mathcal{G}_\theta$ is diagonal for *spatially invariant* operators, and Toeplitz for *periodic pure multiplication* operators.

Example 1: $A = \partial_x$ and $F(x) = \cos(\Omega x)$ have the following representations

$$\mathcal{A}_\theta = \begin{bmatrix} \ddots & & \\ & j\theta + j\Omega n & \\ & & \ddots \end{bmatrix}, \qquad \mathcal{F} = \frac{1}{2} \begin{bmatrix} \ddots & \ddots & & \\ \ddots & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

for every $\theta \in [0,\Omega)$, respectively. Notice that we have dropped the θ subscript in $\mathcal{F}$, as it is independent of this variable. ∎

III. Problem Setup

Let us now consider a system of the form

$$\begin{aligned} \partial_t\, \psi(t,x) &= A\, \psi(t,x) + B\, u(t,x) \\ &= \left(A^{\mathrm{o}} + B^{\mathrm{o}}\, \epsilon\, F\, C^{\mathrm{o}}\right) \psi(t,x) + B\, u(t,x), \\ y(t,x) &= C\, \psi(t,x), \end{aligned} \tag{4}$$

where $t \in [0,\infty)$ and $x \in \mathbb{R}$ with the following assumptions. The (possibly unbounded) operators A^{o}, B^{o}, C^{o} are spatially invariant, and the bounded operators B, C are spatially periodic. $F(x) = 2L\cos(\Omega x)$ with L a constant matrix, and ϵ is a complex scalar. A^{o}, B^{o}, C^{o} and $E := B^{\mathrm{o}}\, F\, C^{\mathrm{o}}$ are all defined on a dense domain $\mathscr{D} \subset L^2(\mathbb{R})$ and are closed. u, y, and ψ are the spatio-temporal input, output, and state of the system, respectively.

Then, as shown in [8] [10] and also briefly in the previous section, the system (4) can be represented as in the Fourier domain as follows.

$$\begin{aligned} \partial_t\, \psi_\theta(t) &= \left(\mathcal{A}^{\mathrm{o}}_\theta + \epsilon\, \mathcal{B}^{\mathrm{o}}_\theta\, \mathcal{F}\, \mathcal{C}^{\mathrm{o}}_\theta\right) \psi_\theta(t) + \mathcal{B}_\theta\, u_\theta(t) \\ &= \left(\mathcal{A}^{\mathrm{o}}_\theta + \epsilon\, \mathcal{E}_\theta\right) \psi_\theta(t) + \mathcal{B}_\theta\, u_\theta(t), \\ y_\theta(t) &= \mathcal{C}_\theta\, \psi_\theta(t), \end{aligned} \tag{5}$$

$\theta \in [0,\Omega)$, where $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ have the form of the operator in (3) and[1]

$$\mathcal{A}^{\mathrm{o}}_\theta = \begin{bmatrix} \ddots & & \\ & A_0(\theta_n) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{B}^{\mathrm{o}}_\theta = \begin{bmatrix} \ddots & & \\ & B^{\mathrm{o}}(\theta_n) & \\ & & \ddots \end{bmatrix}, \quad \mathcal{C}^{\mathrm{o}}_\theta = \begin{bmatrix} \ddots & & \\ & C^{\mathrm{o}}(\theta_n) & \\ & & \ddots \end{bmatrix},$$

$$\mathcal{E}_\theta := \mathcal{B}^{\mathrm{o}}_\theta\, \mathcal{F}\, \mathcal{C}^{\mathrm{o}}_\theta = \begin{bmatrix} \ddots & \ddots & & \\ \ddots & 0 & A_{-1}(\theta_n) & \\ & A_1(\theta_n) & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \tag{6}$$

with $\theta_n := \theta + \Omega n$, $n \in \mathbb{Z}$, and

$$A_1(\,\cdot\,) := B^{\mathrm{o}}(\,\cdot\,)\, L\, C^{\mathrm{o}}(\,\cdot - \Omega), \tag{7}$$

$$A_{-1}(\,\cdot\,) := B^{\mathrm{o}}(\,\cdot\,)\, L\, C^{\mathrm{o}}(\,\cdot + \Omega). \tag{8}$$

We emphasize that the convention used in the representation of $\mathcal{E}_\theta$ in (6) is the same as that used in (3). For example the n^{th} row of $\mathcal{E}_\theta$ is $\{\cdots, 0, A_1(\theta_n), 0, A_{-1}(\theta_n), 0, \cdots\}$.

Remark 1: We note that taking $F(x)$ to be a pure cosine is not restrictive. The results obtained here can be easily extended to problems where $F(x)$ is any periodic function with absolutely convergent Fourier series coefficients. ∎

Remark 2: The system (4) can be considered as the LFT (linear fractional transformation [11]) of the spatially peri-

[1] To avoid clutter, we henceforth drop the "ˆ" from all Fourier symbols.

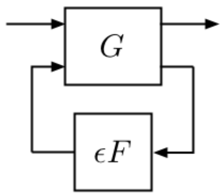

Fig. 2. The LFT of a spatially periodic system G and a spatially periodic multiplication operator ϵF.

odic system G,

$$G = \left[\begin{array}{c|cc} A^{\mathrm{o}} & B & B^{\mathrm{o}} \\ \hline C & 0 & 0 \\ C^{\mathrm{o}} & 0 & 0 \end{array}\right]$$

and the (memoryless and bounded) spatially periodic pure multiplication operator $\epsilon F(x) = \epsilon 2L\cos(\Omega x)$, see Figure 2. ∎

IV. Spectral & Stability Analysis

It is shown in [8] that for a general spatially periodic operator A we have

$$\Sigma(A) \;=\; \overline{\bigcup_{\theta\in[0,\Omega)} \Sigma(\mathcal{A}_\theta)}. \tag{9}$$

In the case where A is spatially invariant (and thus $\mathcal{A}_\theta = \operatorname{diag}\{\cdots, A_0(\theta_n), \cdots\}$), (9) further simplifies to

$$\Sigma(A) \;=\; \overline{\bigcup_{k\in\mathbb{R}} \Sigma_{\mathrm{p}}\big(A_0(k)\big)}. \tag{10}$$

Example 2: Let $A = -(\partial_x^2 + \varkappa^2)^2$. Then $A_0(k) = -(k^2 - \varkappa^2)^2$, see Figure 3 (above). Since $A_0(\,\cdot\,)$ is scalar, $\Sigma_{\mathrm{p}}\big(A_0(k)\big) = A_0(k)$ for every k. It is easy to see that $A_0(\,\cdot\,)$ takes every real negative value and thus from (10) A has *continuous spectrum* $\Sigma(A) = (-\infty, 0\,]$, see Figure 3 (center). ∎

Remark 3: When A is *spatially invariant*, a helpful way to think about $\Sigma(A)$ in terms of its symbol A_0 is suggested by the previous example. First plot $\Sigma_{\mathrm{p}}\big(A_0(\,\cdot\,)\big)$ in the 'complex-plane $\times$ spatial-frequency' space, as in Figure 3 (above) of Example 2. Then the orthogonal projection onto the complex plane of this plot would give $\Sigma(A)$. This can be considered as a geometric interpretation of (10). In Example 2, since $A_0(\,\cdot\,)$ is real scalar and takes only negative values, this projection yields only the negative real axis. But in general if $A_0(\,\cdot\,) \in \mathbb{C}^{q\times q}$, this projection would lead to q curves in the complex plane.

Notice also that in this setting, $\Sigma(\mathcal{A}_\theta)$ is the projection onto the complex plane of samples of $\Sigma_{\mathrm{p}}\big(A_0(\,\cdot\,)\big)$ taken at $k = \theta + \Omega n$, $n \in \mathbb{Z}$, in the 'complex-plane $\times$ spatial-frequency' space. As θ varies in $[0, \Omega)$, these projections trace out $\Sigma(A)$ in the complex plane. This can be considered as a geometric interpretation of (9). Figure 3 (below) shows the said samples in the 'complex-plane $\times$ spatial-frequency' space for a scalar A. ∎

We next introduce a special subclass of *holomorphic* (or *analytic*) semigroups. The reader is referred to [4]–[6] for

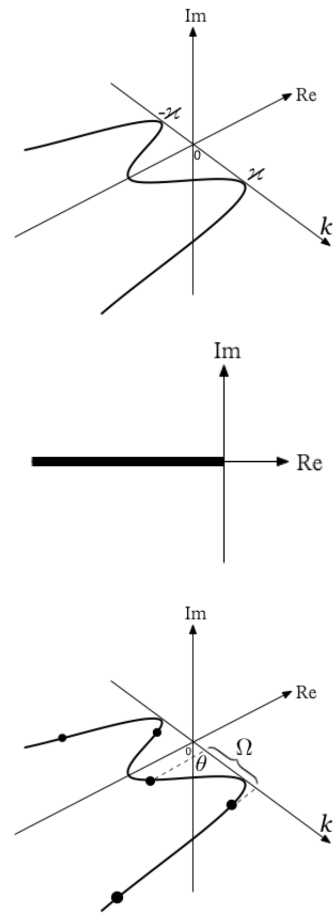

Fig. 3. Above: Representation of the symbol $A_0(\,\cdot\,)$ of Example 2 in 'complex-plane $\times$ spatial-frequency' space. Center: $\Sigma(A)$ in the complex plane. Below: For spatially invariant A, the (diagonal) elements of $\mathcal{A}_\theta$ are samples of the Fourier symbol $A_0(\,\cdot\,)$.

a detailed discussion. Suppose A is densely defined, $\rho(A)$ contains a sector of the complex plane $|\arg(z-\alpha)| \le \frac{\pi}{2}+\gamma$, $\gamma > 0$, $\alpha \in \mathbb{R}$, and there exits some $M > 0$ such that

$$\|(zI - A)^{-1}\| \le \frac{M}{|z-\alpha|} \quad \text{for} \quad |\arg(z-\alpha)| \le \frac{\pi}{2}+\gamma. \tag{11}$$

Then A generates a holomorphic semigroup and we write $A \in \mathscr{H}(\gamma, \alpha, M)$ [6] [4]. We say that A is *sectorial* if A belongs to some $\mathscr{H}(\gamma, \alpha, M)$.

Finally, a semigroup is called exponentially stable if there exist positive constants M and ϱ such that [7]

$$\|e^{At}\| \le M e^{-\varrho t} \quad \text{for} \quad t \ge 0.$$

Theorem 1: Assume that A is sectorial. Then if $\Sigma(A) \subset \mathbb{C}^-$, A generates an exponentially stable semigroup. ∎

Proof: If A is sectorial it defines a holomorphic semigroup, and thus e^{At} is differentiable for $t > 0$ [5]. Then [3] shows that this is sufficient for the spectrum-determined growth condition to hold. In particular, if $\Sigma(A) \subset \mathbb{C}^-$, A generates an exponentially stable semigroup. ∎

V. Stability and the Spectrum-Determined Growth Condition

In the literature on semigroups, there exist examples in which $\Sigma(A)$ lies entirely inside $\mathbb{C}^-$, but $\|e^{At}\|$ does not decay exponetially. See [1] and more recently [2]. In such cases it is said that the semigroup does not satisfy the *spectrum-determined growth condition* [3]. The determining factor in the examples presented in [1] and [2] can be interpreted as the accumulation of the eigenvalues of $\mathcal{A}_\theta$ around $\pm j\infty$ in the form of Jordan blocks of ever-increasing size (i.e. as the eigenvalues tend to $\pm j\infty$ their algebraic multiplicity increases while their geometric multiplicity stays equal to one). But such cases are ruled out when one deals with holomorphic semigroups, which is the reason we consider these semigroups in Theorem 1.

Our ultimate aim in this section is to verify exponential stability. By Theorem 1, in order to prove exponential stability of a holomorphic semigroup with infinitesimal generator A, it is sufficient to show that $\Sigma(A) \subset \mathbb{C}^-$. Hence, in the first part of this section, we give conditions under which the A operators described by (4) do indeed generate holomorphic semigroups. In the second part, we find sufficient conditions which guarantee $\Sigma(A) \subset \mathbb{C}^-$.

Once again, the setup is that of (4). In addition assume that $A_0(k) \in \mathbb{C}^{q\times q}$ is diagonalizable for every $k \in \mathbb{R}$.

Conditions for Sectorial A

To find conditions under which A in (4) will define a holomorphic semigroup we again use perturbation theory. We first find conditions under which A^{o} is sectorial. We then show that $A = A^{\mathrm{o}} + \epsilon\, E$ remains sectorial if E is 'weaker' than A^{o} in a certain sense we will describe and if ϵ is small enough.

In the next theorem we present a condition for a spatially invariant A^{o} with symbol $A_0(k)$ to be sectorial.

Theorem 2: Let $A_0(k)$ be diagonalizable for every $k \in \mathbb{R}$, and let $U(k)$ be the transformation that diagonalizes $A_0(k)$, $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$. Let $\kappa(k) := \|U(k)\|\,\|U^{-1}(k)\|$ denote the condition number of $U(k)$. If $\sup_{k\in\mathbb{R}} \kappa(k) < \infty$, and for every $k \in \mathbb{R}$, $\rho\big(A_0(k)\big)$ contains a sector of the complex plane $|\arg(z-\alpha)| < \frac{\pi}{2} + \gamma$, $\gamma > 0$ and $\alpha \in \mathbb{R}$ both independent of k, then A^{o} is sectorial. ■

Proof: See Appendix. ■

This theorem has a particularly simple interpretation when $A_0(\,\cdot\,)$ is scalar. In this case $\kappa\big(U(k)\big) = 1$ for all $k \in \mathbb{R}$. Now since $A_0(\,\cdot\,)$ traces a curve in the complex plane, by Theorem (2) if this curve stays outside some sector $|\arg(z-\alpha)| \le \frac{\pi}{2} + \gamma$, $\gamma > 0$, of the complex plane then A^{o} is sectorial.

The following theorem is from [6]. It uses the notion of *relative boundedness* of one unbounded operator with respect to another unbounded operator [4].

Theorem 3: Suppose $A^{\mathrm{o}} \in \mathscr{H}(\gamma,\alpha,M)$ and $E = B^{\mathrm{o}}\,F\,C^{\mathrm{o}}$ is relatively bounded with respect to A^{o} so that

$$\|E\,\psi\| \;\le\; a\,\|\psi\| \;+\; b\,\|A^{\mathrm{o}}\,\psi\|, \quad \psi \in \mathscr{D}, \tag{12}$$

with $0 \le a < \infty$ and $0 \le b\,|\epsilon| < 1/(1+M)$. Then $A = A^{\mathrm{o}} + \epsilon\,E$ is a sectorial operator. ■

This theorem says that if A^{o} is sectorial, then so is $A = A^{\mathrm{o}} + \epsilon\,E$ if E is weaker than A^{o} in the sense of (12) and if $|\epsilon|$ is small enough. Notice that at this point, condition (12) can not be reduced to a condition in terms of Fourier symbols (i.e. a condition that can be checked pointwise in k) as in Theorem 2. This is because E is not spatially invariant. But once the exact form of the operators B^{o} and C^{o} is known, (12) *can* be simplified to a condition on the symbols of A^{o}, B^{o} and C^{o}. Let us clarify this statement with the aid of an example.

Example 3: Consider the periodic PDE

$$\begin{aligned} \partial_t\,\psi \;&=\; -(\partial_x^2 + \varkappa^2)^2\,\psi \;-\; c\,\psi \;+\; \epsilon\,\partial_x\,\cos(\Omega x)\,\psi \;+\; u \\ y \;&=\; \psi. \end{aligned}$$

It is easy to see that $A^{\mathrm{o}} = -(\partial_x^2 + \varkappa^2)^2 - c$, $B^{\mathrm{o}} = \partial_x$ and $C^{\mathrm{o}} = \delta(x)$ (the identity convolution operator). By formal differentiation we have

$$E\,\psi \;=\; \partial_x\,\cos(\Omega x)\,\psi \;=\; -\Omega\,\sin(\Omega x)\,\psi + \cos(\Omega x)\,\partial_x\,\psi.$$

Using the triangle inequality and $\|\sin(\Omega x)\| = \|\cos(\Omega x)\| = 1$ we have

$$\|E\,\psi\| \;\le\; |\Omega|\,\|\psi\| \;+\; \|\partial_x\,\psi\|. \tag{13}$$

Thus we have effectively 'commuted out' the (bounded) spatially periodic operator in E, and are left with only spatially invariant operators on the right of (13). Hence, after applying a Fourier transfomation to the right side of (13), a sufficient condition for (12) to hold is that

$$|\Omega| \;+\; |k| \;\le\; a \;+\; b\,|(k^2-\varkappa^2)^2 + c|, \qquad k \in \mathbb{R},$$

which can be shown to be satisfied for large enough $a > |\Omega|$ and $b > 0$. ■

Remark 4: The above example makes clear the notion of E being 'weaker' than A^{o} that we mentioned at the beginning of this subsection. If in Example 3 we had $B^{\mathrm{o}} = \partial_x^{\nu}$ and $C^{\mathrm{o}} = \partial_x^{\mu}$ and $\nu + \mu = 5$, then E would contain a 5th order derivative, whereas the highest order of ∂_x in A^{o} is 4. This would mean that (12) can not be satisfied for any choice of a and b. ■

Conditions for $\Sigma(A) \subset \mathbb{C}^-$

The final step in establishing exponential stability is to show that $\Sigma(A) \subset \mathbb{C}^-$. Unfortunately it is in general difficult to find the spectrum of an infinite-dimensional operator. Thus we proceed as follows. We consider the block-diagonal operators $\mathcal{A}_\theta^{\mathrm{o}}$, $\theta \in [0,\Omega)$. This allows us to extend Geršgorin-type methods (existing for finite-dimensional matrices) to find bounds on the location of $\Sigma(\mathcal{A}_\theta)$, $\mathcal{A}_\theta = \mathcal{A}_\theta^{\mathrm{o}} + \epsilon\,\mathcal{E}_\theta$. This in turn we use to find conditions under which $\Sigma(\mathcal{A}_\theta) \subset \mathbb{C}^-$, and thus $\Sigma(A) \subset \mathbb{C}^-$.

In locating the spectrum of a finite-dimensional matrix $T \in \mathbb{C}^{q\times q}$, the theory of Geršgorin circles [12] provides us with a region of the complex plane that contains the eigenvalues of T. This region is composed of the union of q disks, the centers of which are the diagonal elements of T, and their radii depend on the magnitude of the off-diagonal

elements [12]. This theory has also been extended to the case of finite-dimensional block matrices (i.e., matrices whose elements are themselves matrices) in [13]. Next, we further extend this theory to include bi-infinite (block) matrices $\mathcal{A}_\theta$.

For every $k \in \mathbb{R}$, take $\mathfrak{B}_k$ to be the set of complex numbers z that satisfy

$$\sigma_{\min}\big(zI - A_0(k)\big) \leq \; |\epsilon| \; \Big(\|A_{-1}(k)\| + \|A_1(k)\|\Big), \quad (14)$$

where $\sigma_{\min}\big(zI - A_0(k)\big)$ denotes the smallest singular value of the matrix $zI - A_0(k)$.

Lemma 4: For every θ, the spectrum of $\mathcal{A}_\theta = \mathcal{A}_\theta^{\mathrm{o}} + \epsilon\, \mathcal{E}_\theta$ is contained in the set

$$\mathfrak{S}_\theta \;=\; \overline{\bigcup_{n \in \mathbb{Z}} \mathfrak{B}_{\theta_n}}.$$

Proof: See Appendix. ■

Example 4: Let us consider the periodic PDE [8]

$$\begin{aligned} \partial_t\, \psi \;&=\; -(\partial_x^2 + \varkappa^2)^2\, \psi \;-\; c\, \psi \;+\; \epsilon\, \cos(\Omega x)\, \partial_x\, \psi \;+\; u \\ y \;&=\; \psi. \end{aligned} \quad (15)$$

Comparing (15) and (4) we have

$$\begin{aligned} &A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^{\mathrm{o}}(k) = 1, \quad C^{\mathrm{o}}(k) = jk, \\ &B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}. \end{aligned}$$

From (7)–(8), $A_1(k) = \frac{j}{2}(k - \Omega)$, $A_{-1}(k) = \frac{j}{2}(k + \Omega)$, and thus $\|A_{-1}(k)\| + \|A_1(k)\| = \frac{1}{2}(|k - \Omega| + |k + \Omega|)$. Hence (14) leads to

$$\begin{aligned} \sigma_{\min}\big(zI - A_0(k)\big) = |zI - A_0(k)| &\leq \frac{|\epsilon|}{2}(|k - \Omega| + |k + \Omega|) \\ &= \begin{cases} \Omega\, |\epsilon| & |k| \leq \Omega \\ |k|\, |\epsilon| & |k| \geq \Omega \end{cases}, \end{aligned}$$

which means that the set $\mathfrak{S}_\theta$ is composed of the union of disks with centers at $A_0(\theta_n)$ and radii $\frac{|\epsilon|}{2}(|\theta_n - \Omega| + |\theta_n + \Omega|)$. Figure 4 (above & center) show $\mathfrak{S}_\theta$ in the complex-plane $\times$ spatial-frequency space and in $\mathbb{C}$ respectively.[2] ■

Remark 5: The set

$$\begin{aligned} \Sigma_\varepsilon(M) &:= \{z \in \mathbb{C} \mid \sigma_{\min}(zI - M) \leq \varepsilon\} \\ &\equiv \{z \in \mathbb{C} \mid \|(zI - M)\varphi\| \leq \varepsilon \text{ for some } \|\varphi\| = 1\} \\ &\equiv \{z \in \mathbb{C} \mid z \in \Sigma_{\mathrm{p}}(M + Z) \text{ for some } \|Z\| \leq \varepsilon\} \end{aligned} \quad (16)$$

is called the ε-*pseudospectrum* of the matrix M [14]. Clearly $\Sigma_{\varepsilon'}(M) \subseteq \Sigma_\varepsilon(M)$ if $\varepsilon' \leq \varepsilon$, and $\Sigma_\varepsilon(M) = \Sigma_{\mathrm{p}}(M)$ for $\varepsilon = 0$. The pseudospectrum is composed of small sets around the eigenvalues of M. For instance if M has simple eigenvalues, then for small enough values of ε the pseudospectrum consists of disjoint compact and convex neighborhoods of each eigenvalue [15]. Thus for every $k \in \mathbb{R}$, (14) defines a closed region of $\mathbb{C}$ that includes the eigenvalues of $A_0(k)$. Moreover, comparing (16) and the definition of $\mathfrak{B}_k$ in (14) and we have $\mathfrak{B}_k = \Sigma_\varepsilon\big(A_0(k)\big)$, $\varepsilon = |\epsilon|\, \big(\|A_{-1}(k)\| + \|A_1(k)\|\big)$. ■

[2] We would like to point out that Figure 4 (above) is technically incorrect; once the spatially invariant system is perturbed by a spatially periodic perturbation it is no longer spatially invariant and thus can not be fully represented by a Fourier symbol. Hence its spectrum can no longer be demonstrated in the complex-plane $\times$ spatial-frequency space.

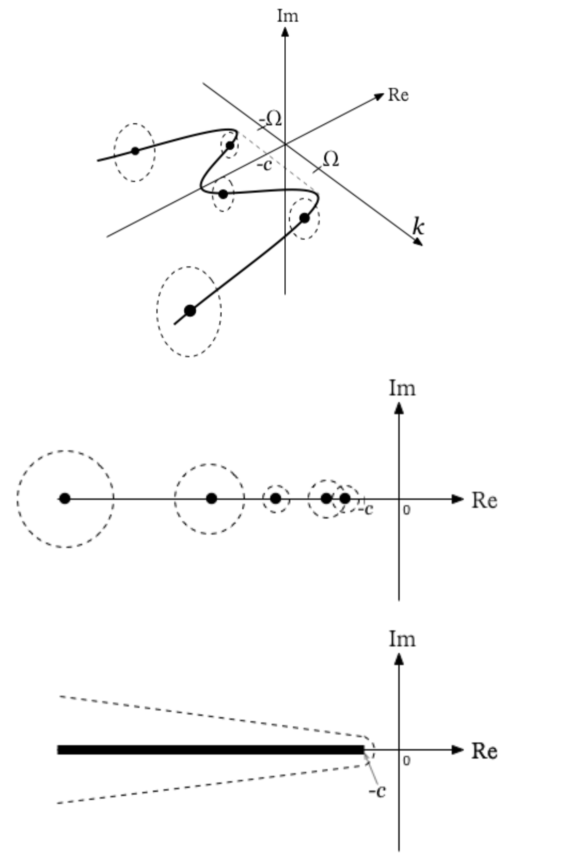

Fig. 4. Above: The $\mathfrak{B}_{\theta_n}$ regions viewed in the 'complex-plane $\times$ spatial-frequency' space (the disks are parallel to the complex plane). Center: $\Sigma(\mathcal{A}_\theta)$ is contained inside the union of the regions $\mathfrak{B}_{\theta_n}$. Below: The bold line shows $\Sigma(A^{\mathrm{o}})$ and the dotted region contains $\Sigma(A)$, $A = A^{\mathrm{o}} + \epsilon E$.

We now employ Lemma 4 to determine whether $\Sigma(A)$ resides completely inside $\mathbb{C}^-$, as needed to assess system stability.

Take $\mathfrak{D}_\varepsilon$ to be the closed disk of radius ε and center at the origin, and $\mathfrak{B}_k$ to be the region described by (14). Define the sum of sets by $\mathfrak{U}_1 + \mathfrak{U}_2 = \{z \mid z = z_1 + z_2, z_1 \in \mathfrak{U}_1, z_1 \in \mathfrak{U}_1\}$. Also, for every $k \in \mathbb{R}$ let $\lambda_{\max}(k)$ represent the eigenvalue of $A_0(k)$ with the maximum real part, and let $\kappa(k)$ be defined as in Theorem 2.

Theorem 5: For every k, $\mathfrak{B}_k$ is contained inside $\Sigma_{\mathrm{p}}(A_0(k)) + \mathfrak{D}_{r(k)}$ with

$$r(k) \;:=\; |\epsilon|\, \big(\|A_{-1}(k)\| + \|A_1(k)\|\big)\, \kappa(k).$$

In particular, if $\Sigma(A^{\mathrm{o}}) \subset \mathbb{C}^-$ and

$$r(k) \;<\; \big|\mathrm{Re}\big(\lambda_{\max}(k)\big)\big| + \beta \quad (17)$$

for every $k \in \mathbb{R}$ and some $\beta < 0$ independent of k, then $\Sigma(A) \subset \mathbb{C}^-$.

Proof: See Appendix. ■

Example 5: Once again we use the scalar system of Example 4. $\kappa(k) = 1$ since $A_0(k)$ is scalar, $|\mathrm{Re}\big(\lambda_{\max}(k)\big)| = |(k^2 - \varkappa^2)^2 + c|$, and

$$\|A_{-1}(k)\| + \|A_1(k)\| \;=\; \frac{1}{2}\,(|k - \Omega| + |k + \Omega|).$$

Thus condition (17) becomes

$$\frac{|\epsilon|}{2}\left(|k-\Omega|+|k+\Omega|\right) \;<\; |(k^2-\varkappa^2)^2+c|+\beta.$$

If this condition is satisfied for some $\beta < 0$, the dotted region in Figure 4 (below) will remain inside $\mathbb{C}^-$ and thus $\Sigma(A) \subset \mathbb{C}^-$. ■

VI. Conclusions and Future Work

In this paper we study the problem of exponential stability for a class of spatially periodic systems. We do this by (i) finding conditions under which the A-operator is sectorial (i.e., generates a holomorphic semigroup) and thus satisfies the spectrum-determined growth condition, and (ii) deriving conditions which guarantee that A has open LHP spectrum.

Future work in this direction would include extending this procedure to larger classes of semigroups which also satisfy the spectrum-determined growth condition.

VII. Appendix

Proof of Theorem 2

It is shown in [16] that a sufficient condition for A^{o} to be sectorial is that $\rho(A^{\mathrm{o}})$ contain some right half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq \mu\}$, and

$$\|z(zI-A^{\mathrm{o}})^{-1}\| \leq M \quad \text{for} \quad \mathrm{Re}(z) \geq \mu,$$

for some $\mu \geq 0$ and $M \geq 1$.

Now since $A_0(k) \in \mathbb{C}^{q\times q}$ has simple eigenvalues for every k, there exists a matrix $U(k)$ such that $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$ with $\Lambda(k)$ a diagonal matrix with elements $\lambda_i(k) := \lambda_i\big(A_0(k)\big)$, $i = 1,\cdots,q$. Thus we have

$$\begin{aligned}
\|z(zI-A^{\mathrm{o}})^{-1}\| \;&\leq\; \sup_{k\in\mathbb{R}}\Big(\|z\big(zI-A_0(k)\big)^{-1}\|\Big)\\
&\leq\; \sup_{k\in\mathbb{R}}\Big(\|U(k)\|\,\|U^{-1}(k)\|\\
&\qquad\qquad\qquad \|z\big(zI-\Lambda(k)\big)^{-1}\|\Big)\\
&=\; \sup_{k\in\mathbb{R}}\Big(\kappa(k)\;\frac{|z|}{\mathrm{dist}[z,\Sigma_{\mathrm{p}}\big(A_0(k)\big)]}\Big)\\
&\leq\; \kappa_{\max}\;\sup_{k\in\mathbb{R}}\Big(\frac{|z|}{\mathrm{dist}[z,\Sigma_{\mathrm{p}}\big(A_0(k)\big)]}\Big),
\end{aligned}$$

where $\kappa_{\max} := \sup_{k\in\mathbb{R}}\kappa(k)$.

Let us now choose $M' = (1+\kappa_{\max})M$, $M > 1$, and consider for a given k the region of the complex plane where

$$\kappa_{\max}\;\frac{|z|}{\mathrm{dist}[z,\Sigma_{\mathrm{p}}\big(A_0(k)\big)]} \;\geq\; M'.$$

This region (which contains the eigenvalues $\lambda_i(k)$) is contained inside the union of the circles

$$\kappa_{\max}\;\frac{|z|}{|z-\lambda_i(k)|} \;\geq\; M', \qquad i = 1,\cdots,q,$$

which are themselves contained inside the larger circles

$$|z-\lambda_i(k)| \;\leq\; \frac{|\lambda_i(k)|}{M}, \qquad i = 1,\cdots,q. \tag{A1}$$

Notice that (A1) describes circles whose radii increase like $|\lambda_i(k)|/M$, $M > 1$, as their centers $\lambda_i(k)$ become distant from the origin. Clearly a sufficient condition for these circles to belong to some open half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) < \mu\}$ for all $k \in \mathbb{R}$ and large enough M is that $\Sigma_{\mathrm{p}}\big(A_0(k)\big)$, $k \in \mathbb{R}$, reside outside some sector $|\arg(z-\alpha)| \leq \frac{\pi}{2}+\gamma$, $\gamma > 0$, of the complex plane.

Finally, if the circles (A1) are contained in some open half plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) < \mu\}$ for all $k \in \mathbb{R}$, then for $\mathrm{Re}(z) \geq \mu$, $z \in \rho(A_0(k))$ and we have

$$\kappa_{\max}\;\sup_{k\in\mathbb{R}}\Big(\frac{|z|}{\mathrm{dist}[z,\Sigma_{\mathrm{p}}\big(A_0(k)\big)]}\Big) \;\leq\; M$$

and thus $\|z(zI-A^{\mathrm{o}})^{-1}\| \leq M$ for $\mathrm{Re}(z) \geq \mu$.

Proof of Lemma 4

We use $\Pi_N\,T\,\Pi_N$ to denote the $(2N+1)\times(2N+1)$ truncation of an operator T on ℓ^2, where Π_N is the projection defined by

$$\mathrm{diag}\Big\{\cdots,0,\underbrace{I,\cdots,I}_{2N+1\text{ times}},0,\cdots\Big\}.$$

where I is the $q\times q$ identity matrix. We form the finite-dimensional matrix $\Pi_N\,\mathcal{A}_\theta\,\Pi_N\big|_{\Pi_N\ell^2}$ with pure point spectrum. Then using a generalized form of the Geršgorin Circle Theorem [13] for finite-dimensional (block) matrices, we conclude that

$$\Sigma\big(\Pi_N\,\mathcal{A}_\theta\,\Pi_N\big|_{\Pi_N\ell^2}\big) \;\subset\; \bigcup_{|n|\leq N}\mathfrak{B}_{\theta_n} \;\subseteq\; \overline{\bigcup_{n\in\mathbb{Z}}\mathfrak{B}_{\theta_n}}$$

where $\mathfrak{B}_{\theta_n}$ are regions of $\mathbb{C}$ defined by (14). Since this holds for all $N \geq 0$, we have $\Sigma(\mathcal{A}_\theta) \subset \mathfrak{S}_\theta$.

Proof of Theorem 5

If $U(k)$ diagonalizes $A_0(k)$, $A_0(k) = U(k)\,\Lambda(k)\,U^{-1}(k)$, and $\kappa(k) = \|U(k)\|\,\|U^{-1}(k)\|$ denotes its condition number, then from [17] the pseudospectrum of $A_0(k)$ satisfies

$$\Sigma_{\mathrm{p}}(A_0(k))+\mathfrak{D}_\varepsilon \;\subseteq\; \Sigma_\varepsilon(A_0(k)) \;\subseteq\; \Sigma_{\mathrm{p}}(A_0(k))+\mathfrak{D}_{\varepsilon\kappa(k)} \tag{A2}$$

for all $\varepsilon \geq 0$. Thus the first statement follows immediately from (A2) and $\mathfrak{B}_k = \Sigma_\varepsilon\big(A_0(k)\big)$ with $\varepsilon = |\epsilon|\,\big(\|A_{-1}(k)\|+\|A_1(k)\|\big)$. To prove the second statement, let $\mathbb{C}^-_\beta$ denote all complex numbers with real part less than $\beta \in \mathbb{R}$. It follows from $\Sigma(A^{\mathrm{o}}) \subset \mathbb{C}^-$ that $\Sigma(\mathcal{A}^{\mathrm{o}}_\theta) \subset \mathbb{C}^-$ for every θ. If (17) holds then

$$\mathfrak{B}_{\theta_n} \;\subseteq\; \Sigma_{\mathrm{p}}(A_0(\theta_n))+\mathfrak{D}_{r(\theta_n)} \;\subset\; \mathbb{C}^-_\beta$$

for every $n \in \mathbb{Z}$, and from Lemma 4 we have $\Sigma(\mathcal{A}_\theta) \subset \mathfrak{S}_\theta \subset \mathbb{C}^-_{\beta'}$ for some $\beta < \beta' < 0$ and every θ. Thus $\Sigma(A) \subset \mathbb{C}^-$.

REFERENCES

[1] J. Zabczyk, "A note on C_0-semigroups," *Bull. Acad. Polon. Sci.*, vol. 23, pp. 895–898, 1975.

[2] M. Renardy, "On the linear stability of hyperbolic PDEs and viscoelastic flows," *Z. angew. Math. Phys. (ZAMP)*, vol. 45, pp. 854–865, 1994.

[3] Z. Luo, B. Guo, and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*. Springer-Verlag, 1999.

[4] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, 1995.

[5] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*. American Mathematical Society, 1957.

[6] M. Miklavčič, *Applied Functional Analysis and Partial Differential Equations*. World Scientific, 1998.

[7] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

[8] M. Fardad, M. R. Jovanović, and B. Bamieh, "Stability, norms and frequency analysis of distributed spatially periodic systems," Tech. Rep. CCEC-05-0720, University of California, Santa Barbara, 2005. http://ccec.mee.ucsb.edu/pdf/ccec-05-0720.pdf.

[9] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *IEEE Transactions Automatic Control*, vol. 47, pp. 1091–1107, July 2002.

[10] M. Fardad and B. Bamieh, "A perturbation approach to the H^2 analysis of spatially periodic systems," in *Proc. ACC*, pp. 4838–4843, 2005.

[11] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Prentice Hall, 1996.

[12] R. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[13] D. G. Feingold and R. S. Varga, "Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem," *Pacific J. Math.*, vol. 12, pp. 1241–1250, 1962.

[14] L. N. Trefethen, "Pseudospectra of matrices," in *Numerical Analysis 1991 (Pitman Research Notes in Mathematics Series, vol. 260)*, pp. 234–266, 1992.

[15] J. V. Burke, A. S. Lewis, and M. L. Overton, "Optimization and pseudospectra, with applications to robust stability," *SIAM J. Matrix Anal. Appl.*, vol. 25, no. 1, pp. 80–104, 2003.

[16] L. Lorenzi, A. Lunardi, G. Metafune, and D. Pallara, *Analytic Semigroups and Reaction-Diffusion Problems*. Internet Seminar, 2005. http://www.fa.uni-tuebingen.de/teaching/isem/2004_05/phase1.

[17] S. C. Reddy, P. J. Schmid, and D. S. Henningson, "Pseudospectra of the Orr-Sommerfeld operator," *SIAM J. Appl. Math.*, vol. 53, no. 1, pp. 15–47, 1993.